

Status of Chiral-Scale Perturbation Theory

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Chiral-scale perturbation theory χPT_σ has been proposed as an alternative to chiral $SU(3)_L \times SU(3)_R$ perturbation theory which explains the $\Delta I = 1/2$ rule for kaon decays. It is based on a low-energy expansion about an infrared fixed point in three-flavor QCD. In χPT_σ , quark condensation $\langle \bar{q}q \rangle_{\text{vac}} \neq 0$ induces nine Nambu-Goldstone bosons: π, K, η and a QCD dilaton σ which we identify with the $f_0(500)$ resonance. Partial conservation of the dilatation and chiral currents constrains low-energy constants which enter the effective Lagrangian of χPT_σ . These constraints allow us to obtain new phenomenological bounds on the dilaton decay constant via the coupling of σ/f_0 to pions, whose value is known precisely from dispersive analyses of $\pi\pi$ scattering. Improved predictions for $\sigma \rightarrow \gamma\gamma$ and the σNN coupling are also noted. To test χPT_σ for kaon decays, we revive a 1985 proposal for lattice methods to be applied to $K \rightarrow \pi$ *on-shell*.

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1. Approximate Scale Invariance in Low-Energy QCD

In the low-energy regime of QCD with heavy quarks t, b, c decoupled, the relevance of scale (dilatation) invariance is determined by the trace anomaly [1]–[4] of the resulting 3-flavor theory:¹

$$\theta_\mu^\mu = \frac{\beta(\alpha_s)}{4\alpha_s} G_{\mu\nu}^a G^{a\mu\nu} + (1 + \gamma_m(\alpha_s)) \sum_{q=u,d,s} m_q \bar{q}q. \quad (1.1)$$

Depending on the infrared behaviour of β , there are only two realistic scenarios (Fig. 1 (A)):

1. If β remains negative and non-zero, possibly diverging linearly at large α_s , scale invariance is explicitly broken by θ_μ^μ being large *as an operator*. There is *no hint* of approximate scale invariance: quantities such as the nucleon mass $M_N = \langle N | \theta_\mu^\mu | N \rangle$ are generated almost entirely by the gluonic term in (1.1). Then conventional chiral $SU(3)_L \times SU(3)_R$ perturbation theory χPT_3 is the appropriate low-energy effective theory for QCD amplitudes expanded in powers of $O(m_K)$ external momenta and light quark masses $m_{u,d,s} = O(m_K^2)$.
2. If β vanishes when α_s runs non-perturbatively to an infrared fixed point α_{IR} , the gluonic term $\sim G_{\mu\nu}^a G^{a\mu\nu}$ in (1.1) is absent and the dilatation current $D_\mu = x^\nu \theta_{\mu\nu}$ becomes conserved in the limit of vanishing quark masses:

$$\begin{aligned} \partial^\mu D_\mu |_{\alpha_s=\alpha_{\text{IR}}} &= \theta_\mu^\mu |_{\alpha_s=\alpha_{\text{IR}}} = (1 + \gamma_m(\alpha_{\text{IR}})) \sum_{q=u,d,s} m_q \bar{q}q \\ &\rightarrow 0, \quad SU(3)_L \times SU(3)_R \text{ limit}. \end{aligned} \quad (1.2)$$

Although the Hamiltonian preserves dilatations in this limit, *the vacuum state is not scale invariant* due to the formation of a quark condensate $\langle \bar{q}q \rangle_{\text{vac}} \neq 0$. As a result, both chiral $SU(3)_L \times SU(3)_R$ and scale symmetry are realized in the Nambu-Goldstone (NG) mode and the spectrum contains nine massless bosons: π, K, η and a 0^{++} QCD dilaton σ . Non-NG bosons remain massive *despite the vanishing of θ_μ^μ* and have their scale set by $\langle \bar{q}q \rangle_{\text{vac}}$. The relevant low-energy expansion involves a combined limit

$$m_{u,d,s} \sim 0 \quad \text{and} \quad \alpha_s \lesssim \alpha_{\text{IR}}, \quad (1.3)$$

and leads to a new effective theory χPT_σ of approximate chiral-scale symmetry [5, 6]. In this scenario, the dilaton mass is set by m_s , so the natural candidate for σ is the $f_0(500)$ resonance, a broad 0^{++} state whose complex pole mass has real part $\lesssim m_K$ [7, 8, 9].

Until now, scenario 1 has been the generally accepted view, but we have observed [5, 6] that χPT_σ offers several advantages over χPT_3 : it explains the mass and width of $f_0(500)$, produces convergent chiral expansions as a result of σ/f_0 being promoted to the NG sector, and most importantly, explains the $\Delta I = 1/2$ rule for non-leptonic K decays (Fig. 1 (B)).

Because approximate scale symmetry is included, the effective Lagrangian for χPT_σ (Sec. 2) contains several new low-energy constants (LECs) yet to be determined precisely from data. Of particular interest is the dilaton decay constant F_σ given by $m_\sigma^2 F_\sigma = -\langle \sigma | \theta_\mu^\mu | \text{vac} \rangle$. If F_σ is roughly 100

¹Here, $G_{\mu\nu}^a$ is the gluon field strength, $\alpha_s = g_s^2/4\pi$ is the strong running coupling, and $\beta = \mu^2 \partial \alpha_s / \partial \mu^2$ and $\gamma_m = \mu^2 \partial \ln m_q / \partial \mu^2$ refer to a mass-independent renormalization scheme with scale μ .

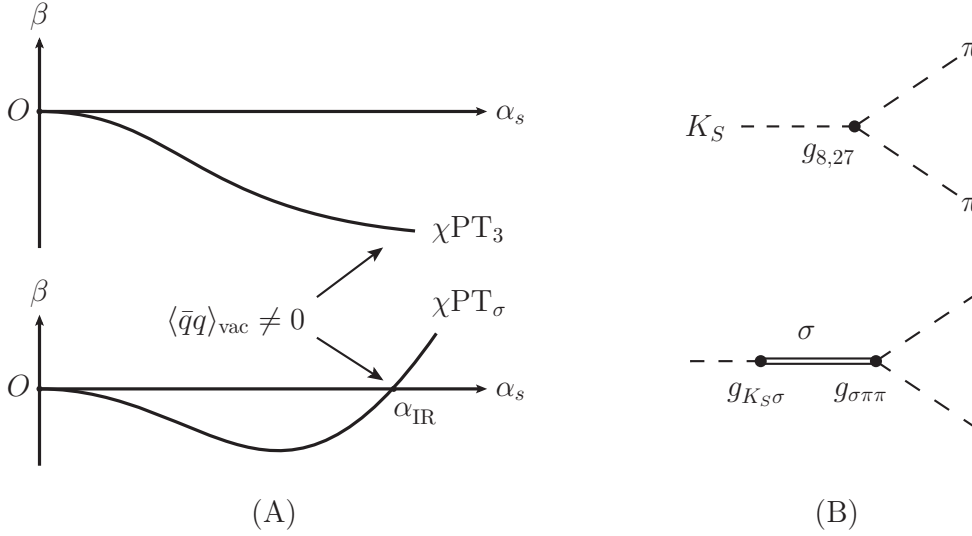


Figure 1: (A) Scenarios for the β function in three-flavor QCD, with corresponding low-energy expansions. In the absence of an infrared fixed point α_{IR} (top diagram), there is no approximate scale invariance and chiral $SU(3)_L \times SU(3)_R$ perturbation theory χPT_3 is relevant at low-energies. If α_{IR} exists (bottom diagram), quark condensation $\langle \bar{q}q \rangle_{\text{vac}} \neq 0$ implies that the NG spectrum contains a QCD dilaton σ , and χPT_3 *must* be replaced by chiral-scale perturbation theory χPT_σ . (B) Diagrams for $K \rightarrow \pi\pi$ decay in lowest-order χPT_σ . The dilaton pole diagram is responsible for the dominant $\Delta I = 1/2$ amplitude.

MeV, scale breaking by the vacuum can generate large masses such as $m_N \approx F_\sigma g_{\sigma NN}$ (Goldberger-Treiman relation for dilatons [10]) for m_σ small. The imprecise value of F_σ in our previous work [5, 6] arose from large uncertainties in the phenomenological value of $g_{\sigma NN}$ [11, 12].

We circumvent this difficulty in Secs. 3 and 4. First, we find new constraints on LECs in the χPT_σ effective Lagrangian by requiring full consistency with the dilatation and chiral currents being conserved in the limit (1.2). These constraints allow us to determine F_σ from the $\sigma\pi\pi$ coupling, whose value is known to remarkable precision from dispersive analyses [7, 8, 9] of $\pi\pi$ scattering. Then we obtain improved predictions for the non-perturbative Drell-Yan ratio

$$R = \sigma(e^+e^- \rightarrow \text{hadrons}) / \sigma(e^+e^- \rightarrow \mu^+\mu^-) \quad \text{at } \alpha_{\text{IR}}, \quad (1.4)$$

as well as the σNN coupling.

In Sec. 5, we resurrect an old proposal [13] to apply lattice QCD for $K \rightarrow \pi$ *on-shell* to determine the couplings $g_{8,27}$ in Fig. 1 (B). Comments on the validity of χPT_σ are reviewed in Sec. 6.

2. Chiral-Scale Lagrangian

For strong interactions, the most general effective Lagrangian of χPT_σ is of the form

$$\mathcal{L}_{\chi\text{PT}_\sigma} = : \mathcal{L}_{\text{inv}}^{d=4} + \mathcal{L}_{\text{anom}}^{d>4} + \mathcal{L}_{\text{mass}}^{d<4} :, \quad (2.1)$$

where

$$d_{\text{anom}} = 4 + \gamma_{G^2}(\alpha_s) \quad \text{and} \quad d_{\text{mass}} = 3 - \gamma_m(\alpha_s) \quad (2.2)$$

are the respective scaling dimensions of $G_{\mu\nu}^a G^{a\mu\nu}$ and $\bar{q}q$. In lowest order (LO) of the chiral-scale expansion, we have $\gamma_m = \gamma_m(\alpha_{\text{IR}})$ and

$$\gamma_{G^2}(\alpha_s) \equiv \beta'(\alpha_s) - \beta(\alpha_s)/\alpha_s = \beta'(\alpha_{\text{IR}}) + O(\alpha_s - \alpha_{\text{IR}}), \quad (2.3)$$

so the resulting terms in (2.1) are

$$\begin{aligned} \mathcal{L}_{\text{inv,LO}}^{d=4} &= \{c_1 \mathcal{K} + c_2 \mathcal{K}_\sigma + c_3 e^{2\sigma/F_\sigma}\} e^{2\sigma/F_\sigma}, \\ \mathcal{L}_{\text{anom,LO}}^{d>4} &= \{(1-c_1) \mathcal{K} + (1-c_2) \mathcal{K}_\sigma + c_4 e^{2\sigma/F_\sigma}\} e^{(2+\beta')\sigma/F_\sigma}, \\ \mathcal{L}_{\text{mass,LO}}^{d<4} &= \text{Tr}(MU^\dagger + UM^\dagger) e^{(3-\gamma_m)\sigma/F_\sigma}, \end{aligned} \quad (2.4)$$

where

$$\mathcal{K} = \frac{1}{4} F_\pi^2 \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) \quad \text{and} \quad \mathcal{K}_\sigma = \frac{1}{2} (\partial_\mu \sigma)^2. \quad (2.5)$$

As $\alpha_s \rightarrow \alpha_{\text{IR}}$, the gluonic anomaly vanishes, so $\mathcal{L}_{\text{anom}} = O(\partial^2, M)$ and we must set $c_4 = O(M)$. Vacuum stability in the σ direction about $\sigma = 0$ (no tadpoles) implies

$$\begin{aligned} 4c_3 + (4 + \beta')c_4 &= -(3 - \gamma_m) \langle \text{Tr}(MU^\dagger + UM^\dagger) \rangle_{\text{vac}} \\ &= -(3 - \gamma_m) F_\pi^2 (m_K^2 + \frac{1}{2} m_\pi^2), \end{aligned} \quad (2.6)$$

so c_3 is also $O(M)$. Expanding (2.4) about $\sigma = 0$ and $U = I$ yields the $\sigma\pi\pi$ coupling

$$\mathcal{L}_{\sigma\pi\pi} = \{[2 + (1 - c_1)\beta'] |\partial\boldsymbol{\pi}|^2 - (3 - \gamma_m) m_\pi^2 |\boldsymbol{\pi}|^2\} \sigma / (2F_\sigma), \quad (2.7)$$

while the corresponding $\sigma\pi\pi$ vertex for an on-shell dilaton is

$$g_{\sigma\pi\pi} = -\frac{1}{2F_\sigma} \left\{ [2 + (1 - c_1)\beta'] m_\sigma^2 + 2[1 - \gamma_m - (1 - c_1)\beta'] m_\pi^2 \right\}. \quad (2.8)$$

3. Effective Energy-Momentum Tensor and its Trace

In any field theory, the energy-momentum tensor can be identified by adding a gravitational source field $g_{\mu\nu}(x)$ coupled to matter fields in a generally covariant fashion. In χPT_σ , this amounts to the substitution

$$\mathcal{L}_{\chi\text{PT}_\sigma}[U, U^\dagger, \sigma] \rightarrow \mathcal{L}_{\chi\text{PT}_\sigma}[U, U^\dagger, \sigma, g_{\mu\nu}], \quad (3.1)$$

where the new effective Lagrangian must be constructed in terms of generally covariant operators. Then the energy-momentum tensor is defined via the variation

$$\theta_{\mu\nu}(x) = 2 \left[\frac{\delta}{\delta g^{\mu\nu}(x)} \sqrt{-g} \mathcal{L}[U, U^\dagger, \sigma, g_{\mu\nu}] \right]_{g_{\mu\nu} = \eta_{\mu\nu}}, \quad (3.2)$$

where $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor and $\eta_{\mu\nu}$ is the flat Minkowski metric. Generalising Donoghue and Leutwyler [14], we obtain the lowest order result

$$\begin{aligned} \theta_{\mu\nu} &= \left[\frac{1}{2} F_\pi^2 \text{Tr}(\partial_\mu U \partial_\nu U^\dagger) - g_{\mu\nu} \mathcal{K} \right] [c_1 e^{2\sigma/F_\sigma} + (1 - c_1) e^{(2+\beta')\sigma/F_\sigma}] \\ &\quad + (\partial_\mu \sigma \partial_\nu \sigma - g_{\mu\nu} \mathcal{K}_\sigma) [c_2 e^{2\sigma/F_\sigma} + (1 - c_2) e^{(2+\beta')\sigma/F_\sigma}] \\ &\quad - g_{\mu\nu} \text{Tr}(MU^\dagger + UM^\dagger) e^{(3-\gamma_m)\sigma/F_\sigma} - g_{\mu\nu} e^{4\sigma/F_\sigma} (c_3 + c_4 e^{\beta'\sigma/F_\sigma}). \end{aligned} \quad (3.3)$$

The trace of (3.3) involves *scale invariant* operators like $\text{Tr}(\partial_\mu U \partial^\mu U^\dagger) e^{2\sigma/F_\sigma}$ which obscure the connection between the scale invariance and a conserved dilatation current D_μ . To remedy this, we “improve” $\theta_{\mu\nu}$ [15] by adding a term

$$I_{\mu\nu} = \frac{F_\sigma^2}{6} (g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) \left[c_2 e^{2\sigma/F_\sigma} + \frac{2(1-c_2)}{2+\beta'} e^{(2+\beta')\sigma/F_\sigma} \right], \quad (3.4)$$

such that the trace of

$$\theta_{\mu\nu}|_{\text{eff}} = \theta_{\mu\nu} + I_{\mu\nu}, \quad (3.5)$$

is given entirely in terms of explicit scale-breaking operators \mathcal{L}_d of scale dimension d :

$$\partial^\mu D_\mu|_{\text{eff}} = \theta_\mu^\mu|_{\text{eff}} = \sum_d (d-4) \mathcal{L}_d. \quad (3.6)$$

Explicitly, the improved trace is

$$\begin{aligned} \theta_\mu^\mu|_{\text{eff}} &= \beta' \mathcal{L}_{\text{anom}}^{d>4} - (1+\gamma_m) \mathcal{L}_{\text{mass}}^{d<4} \\ &= \beta' \{ (1-c_1) \mathcal{K} + (1-c_2) \mathcal{K}_\sigma + c_4 e^{2\sigma/F_\sigma} \} e^{(2+\beta')\sigma/F_\sigma} \\ &\quad - (1+\gamma_m) \text{Tr}(MU^\dagger + UM^\dagger) e^{(3-\gamma_m)\sigma/F_\sigma}. \end{aligned} \quad (3.7)$$

It vanishes in the chiral-scale limit (1.2) only if the low-energy constants associated with $d > 4$ operators satisfy

$$c_1 = c_2 = 1, \quad \text{for } m_{u,d,s} \rightarrow 0 \text{ and } \alpha_s \rightarrow \alpha_{\text{IR}}, \quad (3.8)$$

in addition to the condition $c_4 = O(M)$ required by tadpole cancellation (2.6). Note that the condition $c_1 \rightarrow 1$ in (3.8) ensures that chiral currents have vanishing anomalous dimensions. We can summarise these LO conditions by writing

$$c_i = 1 + O(M), \quad i = 1, 2, \quad (3.9)$$

where the $O(M)$ term is a linear superposition of $O(p^2, M)$ operators and associated LECs.

4. Improved Predictions

An immediate consequence of the constraint (3.9) is that the $\sigma\pi\pi$ coupling for an on-shell dilaton (2.8) takes a particularly simple form

$$g_{\sigma\pi\pi} = -\frac{1}{F_\sigma} [m_\sigma^2 + (1-\gamma_m)m_\pi^2], \quad \text{where } -1 \leq 1-\gamma_m < 2. \quad (4.1)$$

Since the narrow-width approximation is valid in lowest order χPT_σ [6], we have

$$\Gamma_{\sigma\pi\pi} = \frac{|g_{\sigma\pi\pi}|^2}{16\pi m_\sigma} \sqrt{1 - 4m_\pi^2/m_\sigma^2}, \quad (4.2)$$

and this allows us to obtain bounds on F_σ from dispersive analyses of $\pi\pi$ scattering based on the Roy equations. For example, the f_0/σ 's mass and width from [7]

$$m_\sigma = 441_{-8}^{+16} \text{ MeV}, \quad \Gamma_{\sigma\pi\pi} = 544_{-25}^{+18} \text{ MeV}, \quad (4.3)$$

constrain F_σ to lie within the interval $44 \text{ MeV} \leq F_\sigma \leq 61 \text{ MeV}$, where we have allowed $1 - \gamma_m$ to vary according to (4.1). For the moment, we assume that NLO corrections are not a problem.

With F_σ fixed in this manner, we can now use the Golberger-Treiman relation for dilatons [10] to *predict* the value for the σNN coupling. We find $16 \leq g_{\sigma NN} \leq 21$, which is somewhat larger than previous phenomenological determinations [11, 12]. Another important application concerns $\sigma \rightarrow \gamma\gamma$, where an analysis [5, 6] of the electromagnetic trace anomaly in χPT_σ relates the $\sigma\gamma\gamma$ coupling to (1.4):

$$g_{\sigma\gamma\gamma} = \frac{2\alpha}{3\pi F_\sigma} \left(R_{\text{IR}} - \frac{1}{2} \right). \quad (4.4)$$

By fixing $g_{\sigma\gamma\gamma}$ from the di-photon width $\Gamma_{\sigma\gamma\gamma} = 2.0 \pm 0.2 \text{ keV}$ [16], we find $2.4 \leq R_{\text{IR}} \leq 3.1$, which is to be compared with our previous estimate $R_{\text{IR}} \approx 5$ [5, 6].

5. Proposal to test $K \rightarrow \pi$ on the Lattice

The key idea [13] is to keep both K and π on shell and allow $O(m_K)$ momentum transfers.

The lowest-order diagrams for the decay $K \rightarrow \pi\pi$ in Fig. 1 (B) are derived from an effective weak χPT_σ Lagrangian [5, 6]

$$\mathcal{L}_{\text{weak}} = Q_8 \sum_n g_{8n} e^{(2-\gamma_{8n})\sigma/F_\sigma} + g_{27} Q_{27} e^{(2-\gamma_{27})\sigma/F_\sigma} + Q_{mw} e^{(3-\gamma_{mw})\sigma/F_\sigma} + \text{h.c.} \quad (5.1)$$

which reduces to the standard χPT_3 Lagrangian

$$\mathcal{L}_{\text{weak}}|_{\sigma=0} = g_8 Q_8 + g_{27} Q_{27} + Q_{mw} + \text{h.c.} \quad (5.2)$$

in the limit $\sigma \rightarrow 0$. Eqs. (5.1) and (5.2) contain an octet operator [17]

$$Q_8 = J_{13}^\mu J_{\mu 21} - J_{23}^\mu J_{\mu 11}, \quad J_{ij}^\mu = (U \partial^\mu U^\dagger)_{ij} \quad (5.3)$$

the U -spin triplet component [13, 18] of a **27** operator

$$Q_{27} = J_{13}^\mu J_{\mu 21} + \frac{3}{2} J_{23}^\mu J_{\mu 11} \quad (5.4)$$

and a weak mass operator [19]

$$Q_{mw} = \text{Tr}(\lambda_6 - i\lambda_7)(g_M M U^\dagger + \bar{g}_M U M^\dagger). \quad (5.5)$$

Powers of e^{σ/F_σ} are used to adjust the operator dimensions of Q_8 , Q_{27} , and Q_{mw} in (5.1), with octet quark-gluon operators allowed to have differing dimensions at α_{IR} .

In 1985, it was observed [13] that the isospin- $\frac{1}{2}$ term Q_{mw} in Eq. (5.2), when combined with the strong mass term, would be removed by vacuum realignment and therefore could not help solve the $\Delta I = 1/2$ puzzle. In χPT_σ , the outcome is different [5, 6] due to the σ dependence of the Q_{mw} term in Eq. (5.1). Provided there is a mismatch between the weak mass operator's dimension $(3 - \gamma_{mw})$ and the dimension $(3 - \gamma_m)$ of $\mathcal{L}_{\text{mass}}$, the σ dependence of $Q_{mw} e^{(3-\gamma_{mw})\sigma/F_\sigma}$ cannot be eliminated by a chiral rotation. As a result, there is a residual interaction $\mathcal{L}_{K_S\sigma} = g_{K_S\sigma} K_S \sigma$ which mixes K_S and σ in *lowest* $O(p^2)$ order²

$$g_{K_S\sigma} = (\gamma_m - \gamma_{mw}) \text{Re}\{(2m_K^2 - m_\pi^2)\bar{g}_M - m_\pi^2 g_M\} F_\pi / F_\sigma \quad (5.6)$$

²We have corrected a factor of 2 in the formula for the $K_S\sigma$ coupling in our original papers [5, 6].

and produces the $\Delta I = 1/2$ σ -pole amplitude of Fig. 1 (B).

The χPT_3 analysis of 1985 [13] included a suggestion that kaon decays be tested by applying lattice QCD to the weak process $K \rightarrow \pi$, with *both* K and π on shell. It was made at a time when low-lying scalar resonances ($\epsilon(700)$ before 1974, $f_0(500)$ since 1996) were thought not to exist.

This proposal now needs to be taken seriously because:

- Lattice calculations are much easier with only two particles on shell instead of the three in $K \rightarrow \pi\pi$ (all on shell) being analysed by the RBC/UKQCD collaborations [20, 21].
- The 1985 analysis is easily extended to χPT_σ by including σ/f_0 pole amplitudes in chiral Ward identities connecting on-shell $K \rightarrow \pi\pi$ to $K \rightarrow \pi$ on shell. The no-tadpoles theorem

$$\langle K | \mathcal{H}_{\text{weak}} | \text{vac} \rangle = O(m_s^2 - m_d^2), \quad K \text{ on shell}, \quad (5.7)$$

remains valid.

- The lattice result for $K \rightarrow \pi\pi$ on-shell will not distinguish $\Delta I = 1/2$ contributions from the g_8 contact diagram and the σ/f_0 pole diagram in Fig. 1 (B). A lattice calculation of $K \rightarrow \pi$ on shell would measure g_8 (and g_{27}) directly, with no interference from σ/f_0 poles. Then we would *finally* learn whether g_8 is unnaturally large or not.

A key feature of the proposal is that the operator in the on-shell amplitude $\langle \pi | [F_5, \mathcal{H}_{\text{weak}}] | K \rangle$ necessarily carries *non-zero* momentum $q^\mu = O(m_K)$. For either χPT_σ or χPT_3 , the $K \rightarrow \pi$ amplitude can be evaluated in the range

$$-m_K^2 \lesssim q^2 \leq (m_K - m_\pi)^2. \quad (5.8)$$

We highlight the point $q^\mu \neq 0$ because since 1985, there has been a widespread misconception in the literature³ that the analysis [13] involved setting $q^\mu = 0$ as in [19], with the pion in $K \rightarrow \pi$ sent off shell via an interpolating operator. There was and is no reason for this. For example, when writing a soft meson theorem for $\Sigma \rightarrow p\pi$, it is not necessary to force one of the baryons off shell.

6. Issues

When considering the validity of χPT_σ , it is important to avoid any presumption that dimensional transmutation necessarily implies that θ_μ^μ is large and $\neq 0$. Implicit in this intuition is a prejudice that scale invariance cannot be strongly broken via the vacuum when $\theta_\mu^\mu \rightarrow 0$. If the dilaton is a true NG boson, i.e. $m_\sigma \rightarrow 0$ with $F_\sigma \neq 0$ for $\theta_\mu^\mu \rightarrow 0$, it can couple to mass insertion terms in Callan-Symanzik equations and cause them to be *non-zero* in the zero-mass limit. Then Green's functions do not exhibit the power-law scaling expected for manifestly scale-invariant field theories.

This point is illustrated for the quark condensate in Fig. 1 (A). In scenario 1 (top diagram), the running of α_s is driven by the presence of quantities like $\langle \bar{q}q \rangle_{\text{vac}}$ (a mechanism often cited in papers on walking gauge theories [22]). In scenario 2 (bottom diagram), the running coupling freezes at α_{IR} , where the condensate is a *scale-breaking property of the vacuum*.

³We thank the final referee of our long paper [6] for drawing our attention to this.

Lattice investigations of IR fixed points inside the conformal window $8 \lesssim N_f \leq 16$ all depend on naive scaling of Green's functions [22], so they correspond to *scale-invariant vacua*. A recent lattice study [23] of the running of α_s for two flavors with *no* naive scaling suggests that it freezes: the fixed point realises scale invariance in NG mode, i.e. with a scale-breaking vacuum. That is what χPT_σ assumes for three flavors.

The term “dilaton” often refers to a spin-0 particle or resonance which couples to $\theta_{\mu\nu}$ and acquires its mass “spontaneously” due to self interactions. Originally, this idea concerned a scalar component of gravity [24], but now it is a key ingredient of dynamical electroweak symmetry breaking (pp. 198 and 1622-3, PDG tables [9]). This approximates theories with *scale-invariant vacua*, as is evident in walking technicolor. Therefore it has *nothing* to do with our dilaton [25].

It is well known that a resonance cannot be represented by a local interpolating operator, so is the fact that $\sigma/f_0(500)$ has a finite width a problem for χPT_σ ? The answer is “no” because χPT_σ is an expansion in powers and logarithms of $m_{\pi,K,\eta,\sigma}$ with coefficients determined in the *exact* chiral-scale limit (1.2) where σ has zero width [6]. In any perturbation theory, decay rates are calculated that way.

A related remark concerns what is current best practice for scenario 1. The resonance $f_0(500)$ is treated as a member of the non-NG sector with an accidentally small mass. It causes χPT_3 to produce divergent expansions for amplitudes involving $f_0(500)$ poles: the radius of convergence is too small. Instead, these amplitudes are approximated dispersively via contributions from the dominant $f_0(500)$ poles with corrections from nearby thresholds, subject to exact chiral $SU(3) \times SU(3)$ constraints such as Adler zeros. One would certainly not use local fields in this framework.

However χPT_σ is a more ambitious theory. Having promoted σ/f_0 to the NG sector, we expect convergent asymptotic expansions for *all* mesonic amplitudes (scenario 2). The NLO corrections are still being worked out, but a first guess is to set all multi-dilaton vertices to zero. That is equivalent to adding the simplest dilaton diagrams to all χPT_3 diagrams. It seems to produce amplitudes very similar to those of the dispersive approximations of scenario 1.

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